3. Bautin, N. N., Behavior of Dynamic Systems near the Boundaries of the Stability Domain. M.-L., Gostekhizdat, 1949.

Translated by B. D.

ON THE THEORY OF TWO-WAY TRAFFIC FLOW

PMM Vol. 35, №5, 1971, pp. 941-944
G. P. SOLDATOV
(Saratov)
(Received January 28, 1971)

The propagation of a shock wave, the process of decay of the wave, and the motion of a type of simple wave in a two-way uniform symmetric traffic flow are studied on the bases of the hydrodynamic model postulated by J. Blick and G. Newell [1]. An expression is obtained that connects the flow parameters at the front of the shock wave. The decay of the wave is investigated in the vicinity of the head part by using expressions of the parameters in power series in a small quantity. Terms of the expansions are calculated that characterize the rate of change of the wave profile and its curvature at the wave front.

1. Formulation of the problem. In the hydrodynamic theory of two-way uniform symmetric traffic flow two continuity equations

$$\frac{\partial p}{\partial t}+\frac{\partial (pu)}{\partial x}=0, \qquad \frac{\partial q}{\partial t}+\frac{\partial (qv)}{\partial x}=0 \tag{1.1}$$

are used to determine the form of dependence on the independent variables of the average speeds u and v and densities p and q of two homogeneous streams of traffic moving in opposite directions. The system of equations (1.1) is completed by the two empirical relations

$$u=v_0-\alpha p-\beta q, \qquad v=u_0+\beta p+\alpha q \tag{1.2}$$

which represent the average speeds as functions of the densities of both streams.

The region of physically acceptable solutions of the system (1.1), (1.2) is bounded; it can be represented as the union of the regions of hyperbolicity and ellipticity of the system of equations [1, 2]. The laws of motion and growth of initially small perturbations of the flow parameters and the magnitude of the time interval required for the transformation of a weak discontinuity into a shock wave are obtained in [3].

On the generated shock wave the following conditions are satisfied [1] :

$$\begin{aligned} p_0\,[u\,(p_0,\,q_0)-w] &= p\,[u\,(p,\,q)-w] \\ q_0\,[v\,(p_0,\,q_0)-w] &= q\,[v\,(p,\,q)-w] \end{aligned} \tag{1.3}$$

Here the initial unperturbed state is denoted by subscript zero and w, denotes the speed of the shock wave. Eliminating w we obtain the equation of the shock polar

$$\frac{pu\,(p,\,q)-p_0u\,(p_0,\,q_0)}{p-p_0}=\frac{qv\,(p,\,q)-q_0v\,(p_0,\,q_0)}{q-q_0} \tag{1.4}$$

2. Motion of the shock wave. In equations (1.1), (1.2) we introduce dimensionless quantities according to the equations [1]

$$p' = \frac{\alpha p}{u_0}, \qquad q' = \frac{\alpha q}{u_0}, \qquad u' = \frac{u}{u_0}, \qquad v' = \frac{v}{u_0}$$
$$w' = \frac{w}{u_0}, \qquad \beta' = \frac{\beta}{\alpha}, \qquad t' = u_0 t$$

and obtain the following system of equations, which it is convenient to write in matrix form [3], with the primes omitted

$$U_t + AU_x = 0, \qquad U = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad A = \begin{bmatrix} (pu)_p & pu_q \\ qv_p & (qv)_q \end{bmatrix} \tag{2.1}$$

$$u = 1 - p - \beta q, \qquad v = -1 + \beta p + q \tag{2.2}$$

The equation of the shock polar is not changed by making the change to new variables. We rewrite the system (2.1) in the form

$$L^{1,2}\,[U_t + \lambda^{1,2} U_x] = 0 \tag{2.3}$$

Here

$$\lambda^{1,2} = [(\beta - 2)(p - q) \pm R]/2$$
$$(R = ([2 - (\beta + 2)(p + q)]^2 - 4\beta^2 pq)^{1/2})$$
$$L^{1,2} = [2 - (\beta + 2)(p + q) \pm R - 2\beta p]$$

are respectively the eigenvalues and the left eigenvectors of the matrix A.

In the hodograph plane along the characteristics $\Gamma^{1,2}$, whose equation has the form

$$\beta p\,(dq)^2 - [2 - (\beta + 2)(p + q)]\,dp\,dq + \beta q\,(dp)^2 = 0 \tag{2.4}$$

the system of equations (2.3) is written as

$$L^{1,2} dU = 0, \qquad dU = \begin{bmatrix} dp \\ dq \end{bmatrix} \tag{2.5}$$

We substitute the expressions (2.2) for the average speeds into the Eq. (1.4) of the shock polar and differentiate the result with respect to p. Thus we obtain

$$P\left\{(1 + \beta)\,Q^2 P - \frac{\beta p_0}{q_0} P^2 + \beta Q^2\right\} dq + Q\left\{(1 + \beta)\,QP^2 - \frac{\beta q_0}{p_0} Q^2 + \beta P^2\right\} dp = 0 \tag{2.6}$$

Here

$$P = (p - p_0)\,/\,p_0, \qquad Q = (q - q_0)\,/\,q_0$$

denote the intensities of the shock waves for each of the one-way streams forming the two-way flow. The expression (2.6) together with Eq. (2.5) along the C^1 characteristics

$$[2 - (\beta + 2)(p + q) + R]\,dp - 2\beta p\,dq = 0$$

shows the behavior of the flow parameters at the shock wave front in its motion in the two-way stream.

3. Decay of the wave. In Eqs. (2.1), (2.2) we change to the variables t' and τ using the equations

$$t' = t, \qquad \tau = t - \frac{x}{\lambda}, \qquad \lambda = \frac{dx}{dt}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = -\frac{1}{\lambda}\frac{\partial}{\partial \tau}$$

and represent the unknown densities p and q of the flow in the vicinity of the front $\tau = 0$ as power series in τ

$$p = \sum_{k=0}^{\infty} p_k(t)\,\tau^k, \qquad q = \sum_{k=0}^{\infty} q_k(t)\,\tau^k \tag{3.1}$$

Substituting (3.1) into the transformed equations and equating coefficients of like powers of τ, we obtain the equation of the zeroth approximation

$$[1 - 2p_0 - \beta q_0 - \lambda]\, p_1 = \beta p_0 q_1 \tag{3.2}$$

the equations of the first approximation

$$[1 - 2p_0 - \beta q_0 - \lambda]\, p_2 = \beta p_0 q_2 + {}^1/_2\,[\lambda p_1^{\cdot} + 2p_1(p_1 + \beta q_1)] \tag{3.3}$$

$$\frac{1 - 2p_0 - \beta q_0 - \lambda}{\beta q_0} = \frac{\lambda p_1^{\cdot} + 2p_1(p_1 + \beta q_1)}{\lambda q_1^{\cdot} - 2q_1(\beta p_1 + q_1)} \tag{3,4}$$

and the equations of the second approximation

$$[1 - 2p_0 - \beta q_0 - \lambda]\, p_3 = \beta p_0 q_3 + {}^1/_3\,[\lambda p_2^{\cdot} + 3(2p_1 + \beta q_1)\,p_2 + 3\beta p_1 q_2]$$

$$\frac{1 - 2p_0 - \beta q_0 - \lambda}{\beta q_0} = \frac{\lambda p_2^{\cdot} + 3(2p_1 + \beta q_1)\,p_2 + 3\beta p_1 q_2}{\lambda q_2^{\cdot} - 3\beta q_1 p_2 - 3(\beta p_1 + 2q_1)\,q_2} \tag{3.5}$$

In these equations dots in the positions of the primes denote derivatives with respect to time. The initial conditions are

$$t = 0, \qquad p_1 = p_{10}, \qquad q_1 = q_{10}$$

The solution of Eqs. (3.2) and (3.4)

$$q_1 = \frac{q_0}{p_0}\,\mu p_1$$

$$p_1 = \lambda p_{10}\left[1 - \frac{q_0}{p_0}\,\mu^2\right]\left[B + \lambda\left(1 - \frac{q_0}{p_0}\,\mu^2\right)\right]^{-1}$$

$$\mu = [1 - 2p_0 - \beta q_0 - \lambda]\,/\,\beta q_0$$

$$B = 2p_{10}\left\{(1 + \beta)\left(1 + \frac{q_0}{p_0}\,\mu\right) - \left(\beta + \frac{q_0}{p_0}\,\mu\right)\left(1 - \frac{q_0}{p_0}\,\mu^2\right)\right\} t$$

expresses the law of decay of the rate of change of the wave front.

The solution of Eqs. (3.3) and (3.5)

$$q_2 = \frac{q_0}{p_0}\,\mu p_2 + D\beta p_0\left(1 - \frac{q_0}{p_0}\,\mu^2\right) p_1^2$$

$$p_2 = \left(\frac{p_1}{p_{10}}\right)^3 (c p_{10}^3 t + p_{20})$$

$$c = -D\,[2\mu E + 3\,(1 - q_0\mu^2/p_0)(\beta + \mu\beta + 2q_0\mu^2/p_0)]$$

$$D = (1 + \beta)\,(1 + q_0\mu/p_0)\,q_0\mu^2/\beta\,(p_0 - q_0\mu^2)^2$$

$$E = 2\,[(1 + \beta)\,(1 + q_0\mu/p_0) - (\beta + q_0\mu/p_0)\,(1 - q_0\mu^2/p_0)]$$

characterizes the decay of the curvature of the shock front.

4. A particular solution. We will seek a solution of Eqs. (2.1) and (2.2) in the form of a simple wave, that is, we assume that $p = p(q)$. The system (2.1), (2.2) is then reduced to one equation

$$q_t + [\beta q p^{\cdot} - 1 + \beta p + 2q]\, q_x = 0 \tag{4.1}$$

and the compatibility condition (2.4) for the equation under consideration. The solution of Eq. (4.1)

$$x - [\beta q p^{\cdot} - 1 + \beta p + 2q]\, t = f(q)$$

contains an arbitrary function $f(q)$, whose form is determined by the pattern of the initial distribution of density in one of the one-way streams.

The solution of the compatibility equation (2.4), written in parametric form [1]

$$p = r\frac{ds(r)}{dr} - s(r), \qquad q = \frac{ds(r)}{dr}, \qquad \alpha = \frac{\beta}{2(1+\beta)}$$

$$s(r) = \frac{r^{\alpha}(1+r)^{1-2\alpha}}{1+\beta}\left[\int \frac{dr}{r^{\alpha}(1+r)^{2-2\alpha}} + c\right]$$

together with the particular solution $p + q = 1$ represents the desired relation.

The author thanks S.V. Fal'kovich for discussion of the results of this work.

BIBLIOGRAPHY

1. Bick, J.H. and Newell, G.F., A continuum model for two-directional traffic flow. Quart. Appl. Math., Vol.18, №2, 1960.

2. Fortet, R., La circulation des voitures sur une route ou dans les arteres d'une ville, a partir des lois de l'hydraulique. Houille Blanche, 1963, Vol.18, №8, p.909 - 914.

3. Soldatov, G.P., The instant of formation of a shock wave in a two-way traffic flow. PMM, Vol.34, №1, 1970.

Translated by M.D.V.D.